Probing an untouchable environment as a resource for quantum computing

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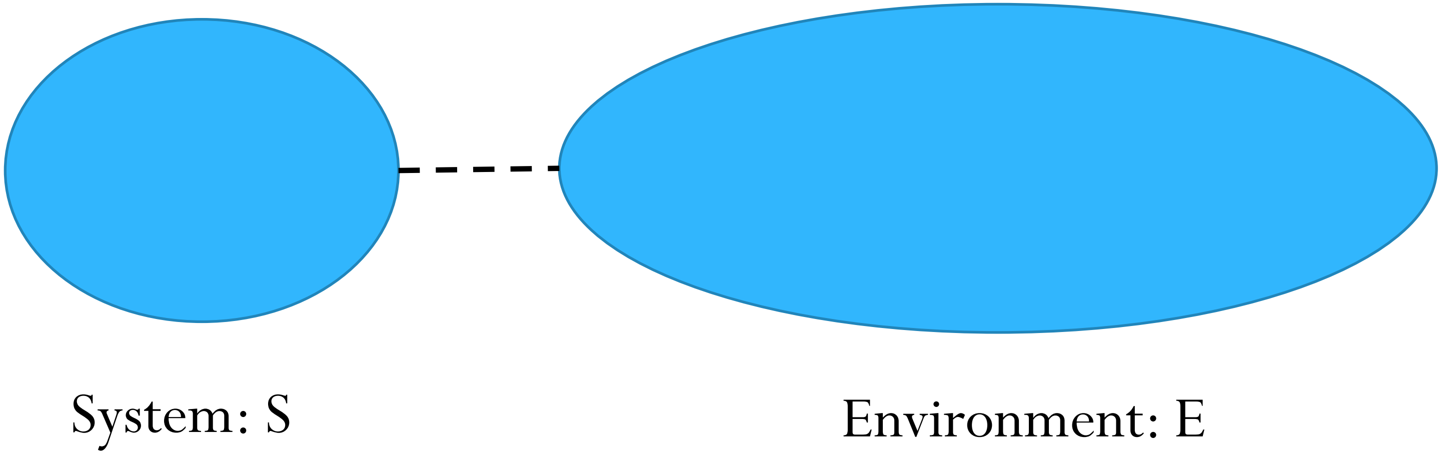
arXiv:1301.2152

Table of Contents

1. Introduction

Introduction

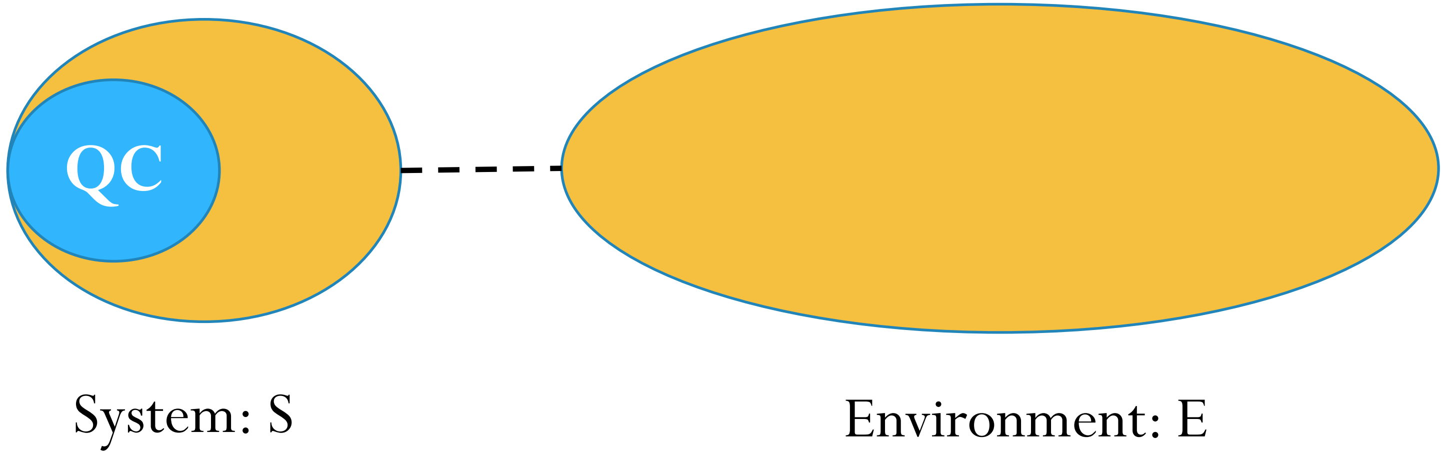
Quantum dynamics is always fraught with decoherence.



The interactions with the environment that cause untraceable and uncontrollable loss of quantum information.

Introduction

Decoherence ⟺ Quantum Error Correction Code



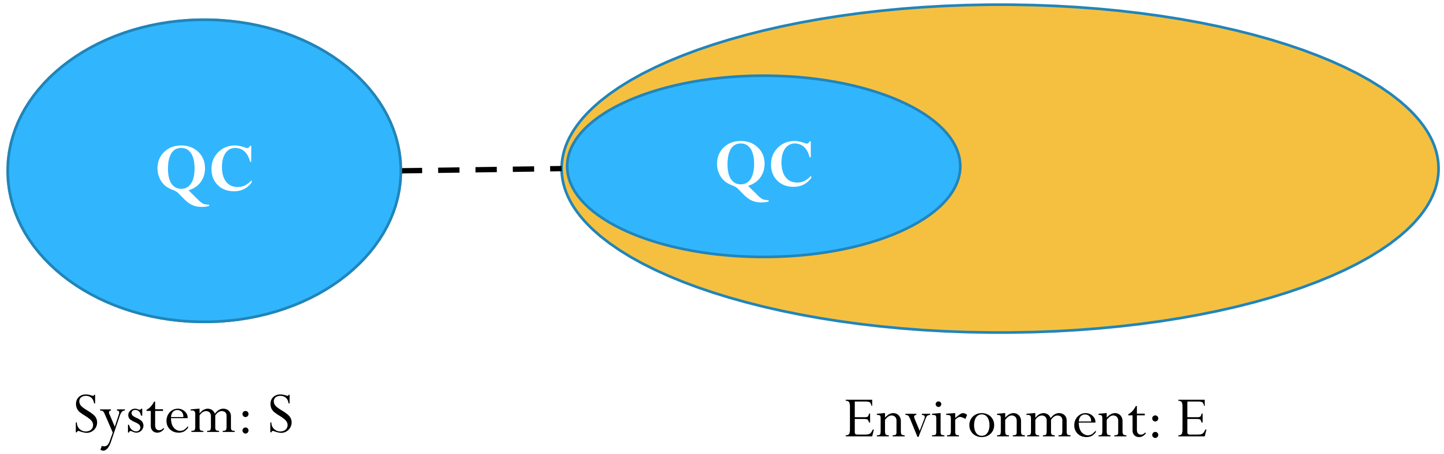
As a result, the size of the system available for quantum computation (QC) is effectively much smaller than the system S.

Question: Is this discussion (QC < S) always true?

Introduction

Question: Can we make QC > S ?

Can we use a part of E as QC?



Answer: Impossible! for Markovian case.

Markovian: the quantum information never returns from E.

Then, how about non-Markovian case?

Introduction

Question: Can we make QC > S ?

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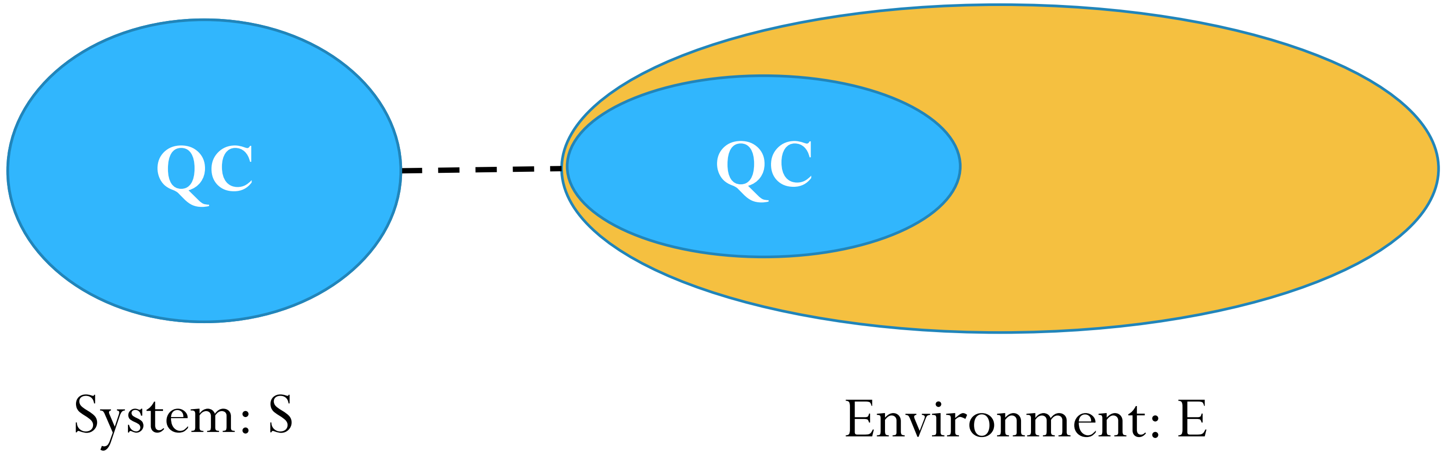


Answer: Possible for Non-Markovian (or general) case.

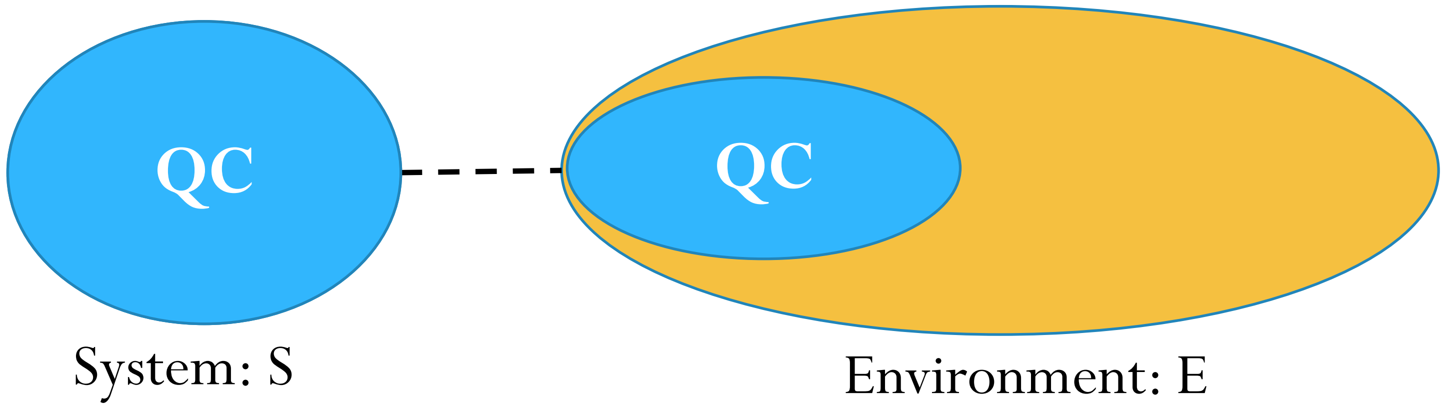
In general, because of unitary evolution of the joint system SE:

the quantum information may return from E.

Introduction

Idea: The joint system S+E is described by

State on SE: $|\Psi_{SE}\rangle$ and Interaction Hamiltonian: H_{SE}



If we can identify $|\Psi_{SE}\rangle$ and H_{SE}, a part of E can be used for QC, by an indirect control.

Let's assume $|\Psi_{SE}\rangle$ and H_{SE} are known.

Let's see how to control a part of E through H_{SE}.

2. Quantum Control Theory

For detail of the quantum control theory, see D. D'Alessandro "Introduction to Quantum Control and Dynamics" , Taylor and Francis, Boca Raton, (2008)

Quantum control theory (open loop)

Classical control systems:

$$\frac{d}{dt}|\psi(t)\rangle = iH\big(f(t)\big)|\psi(t)\rangle$$

$|\psi(t)\rangle$: a (pure) state on a Hilbert space $\mathcal{H}$.

$H\big(u(t)\big)$: a time dependent Hamiltonian

$f(t)$: control parameters, e.g. external fields

Question:

- Which $|\psi(t)\rangle$ and/or unitary are reachable? (controllability)
- How quick they are reachable? (optimality)

Quantum control theory (open loop)

Operator controllability :

$$\frac{d}{dt}V(t) = iH\big(f(t)\big)V(t) \quad -(*)$$

Def.: the system is controllable,

if $\forall V \in U(d), \exists f(t), t_0,$ s.t. $V = V(t_0)$ and $V(t)$ is a solution of (*) with $V(0) = I$.

In other words, the system is controllable if all unitary is reachable by properly choosing parameters $f(t)$.

Quantum control theory

Operator controllability :

$$\frac{d}{dt}V(t) = iH\big(f(t)\big)V(t) \quad -(*)$$

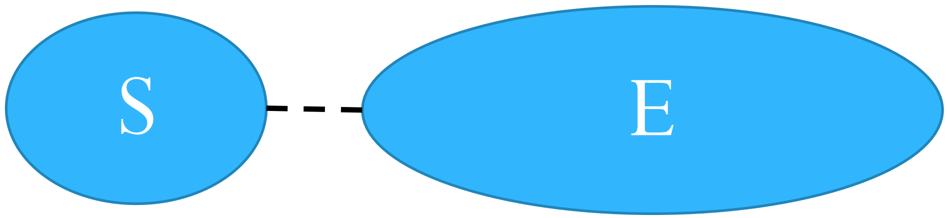
Theorem: the system is controllable, if and only if
Dynamical Lie group e^L is equal to $U(d)$, where $d = \dim \mathcal{H}$.

Dynamical Lie algebra: $L := L\{iH(f): \forall f\}$

Dynamical Lie group: e^L := Lie group corresponding to L.

We can operate all unitary operations in e^L by modulating $f(t)$.

Quantum control theory



Our control system:

$$H(t) = H_{SE} + \sum_{m=1}^{M} f_m(t) H_S^{(m)} \otimes I_S,$$

$f_m(t)$: control amplipudes

$\left\{H_S^{(m)}\right\}_m$: a basis of algebra $u(d_S)$.

$$\mathrm{u}(\mathrm{d}_S) = L\left\{iH_S^{(m)} \otimes I_E : m\right\} \subset L\left\{iH_{SE}, iH_S^{(m)} \otimes I_E : m\right\}.$$

Available unitary operations are more than $U(d_S)$.

Question: How large $L\left\{iH_{SE}, iH_S^{(m)} \otimes I_E : m\right\}$ is?

Quantum control theory

Question: How large $L\left\{iH_{SE}, iH_S^{(m)} \otimes I_E : m\right\}$ is?

Answer: of course depends on H_{SE}. But…

<Quantum universal interface> (S. Lloyd et al PRA 2003)

When S is a qubit, and

$H_{SE} = \sigma_z \otimes A + I_S \otimes H$ with generic A and H,

the whole system is controllable:

$$L\left\{iH_{SE}, iH_S^{(m)} \otimes I_E : m\right\} = u(d_S d_E). -(**)$$

(**) also holds in many other cases.

∴ E can be used as QC, when $|\Psi_{SE}\rangle$ and H_{SE} are known.

Quantum control theory

$$L\left\{iH_{SE}, iH_S^{(m)} \otimes I_E : m\right\} = u(d_S d_E)$$

also holds in many cases including spin chains:

- S. Schirmer, Pullen, Pemberton-Ross PRA 2008
- Burgarth, Giovannetti PRA 2009

etc.…

$\therefore$ E can be used as QC, when $|\Psi_{SE}\rangle$ and H_{SE} are known.

Question: Can we know $|\Psi_{SE}\rangle$ and H_{SE}?

= Can we perform tomography of an environment?

……there are known partial results…

3. Tomography of environment

Tomography of environment

Question: Can we know $|\Psi_{SE}\rangle$ and H_{SE}?

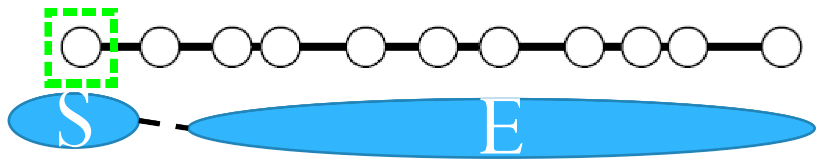
$=$ Can we perform tomography of an environment?

……there are known partial results…

In many body systems described by Spin Chains etc,

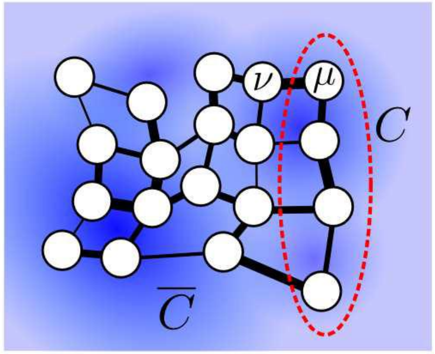
Hamiltonian can be identified under the condition of limited access.

D. Burgarth, K. Maruyama, F. Nori, Phys. Rev. A 79, 020305(R) (2009)



D. Burgarth, K. Maruyama, New J. Phys. 11 (2009) 103019

$$H = \sum_{(m,n)\in E} c_{mn}\left(\sigma_m^x\sigma_n^x + \sigma_m^y\sigma_n^y + \Delta\sigma_m^z\sigma_n^z\right) + \sum_{n\in V} b_n\sigma_n^z,$$

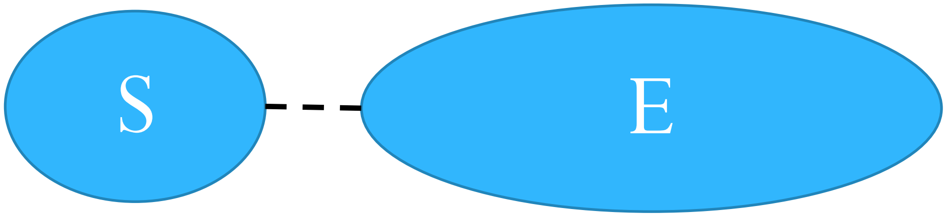


However, they assume a prior knowledge of H_{SE}.

Tomography of environment

Question: Suppose we do not know anything about the environment .

Can we tomography $|\Psi_{SE}\rangle$ and H_{SE} by controlling the system S?



Answer: In Daniel and Maruyama's problem settings, No!

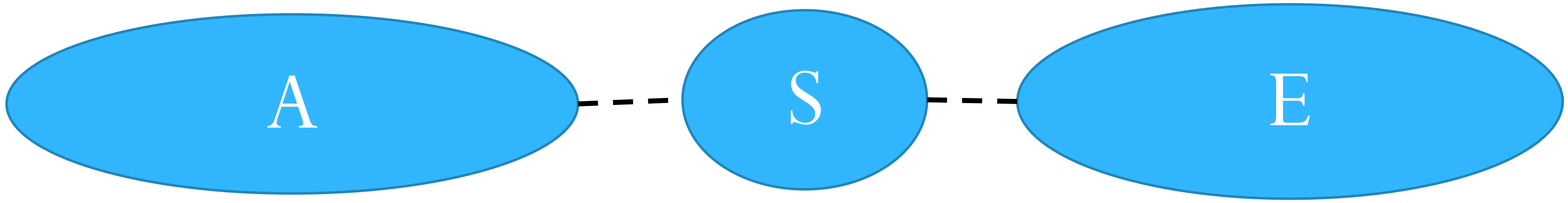
We can easily find an example from their models.

We need to increase our capability.

Tomography of environment

Our problem settings

1. $\dim \mathcal{H}_S < +\infty$ and $\dim \mathcal{H}_E < +\infty$
2. An arbitrarily large ancilla (the system A) is available.
3. We can instantaneously implement any quant. operations on AS.
4. The joint system can be prepared in $|\Psi_{ASE}\rangle$ at an initial time t_0.



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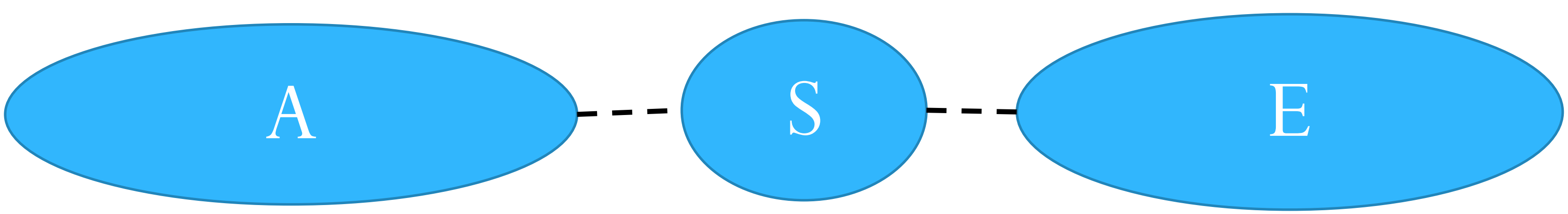
Necessary: "Quantum information always return from E." .

S effectively interacts with only a finite dimensional subspace E of the *universe* E'



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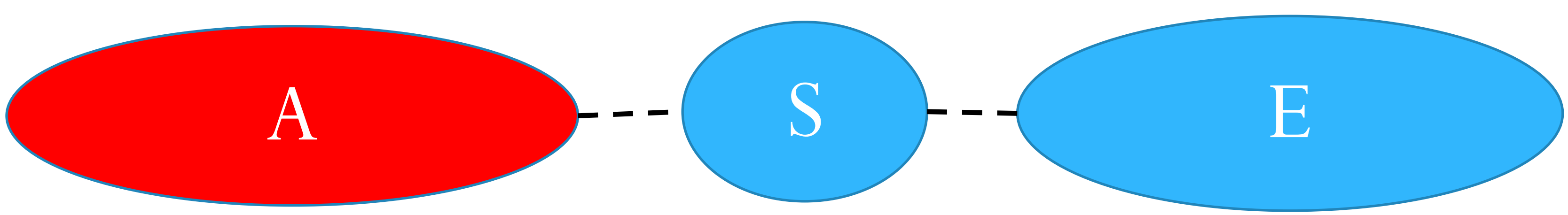
Assume: A does not directly couple to E, and universal comp. on A.

NTT has very expensive QC A, and want to sell SE.



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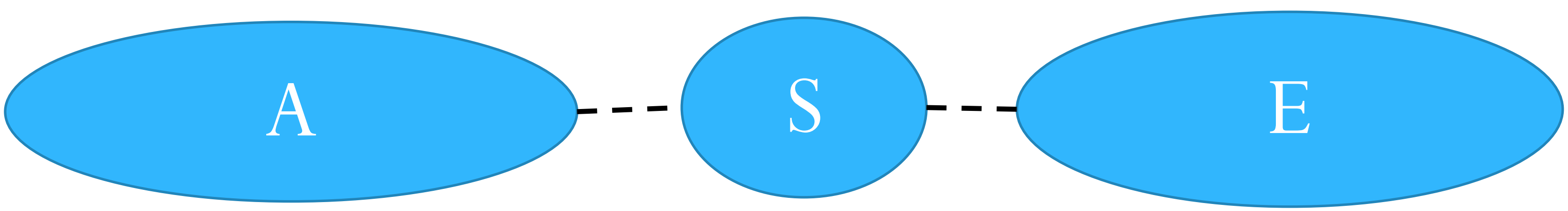
4. The joint system can be prepared in $|\Psi_{ASE}\rangle$ at an initial time t_0.

Assume: $|\Psi_{ASE}\rangle$ is an unknown but identical state.

Note: we can assume ASE is in pure at initial time: ASE can be purify by adding E_2. Then, redefine EE_2 as a new E.



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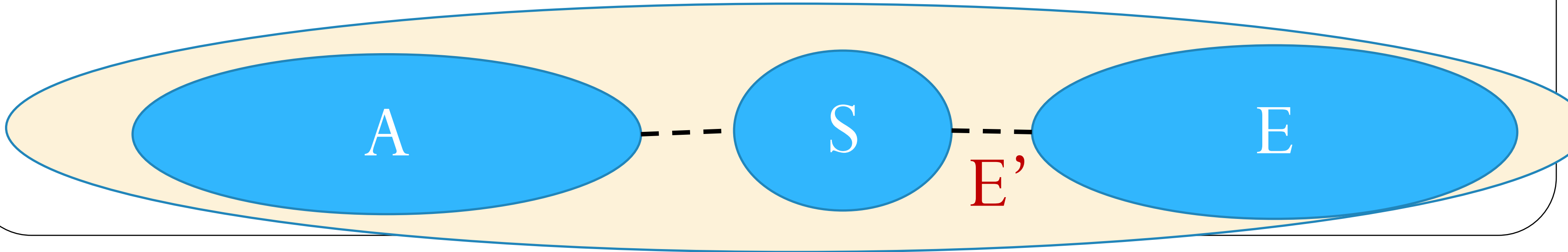
4. The joint system can be prepared in $|\Psi_{ASE}\rangle$ at an initial time t_0.

In practice, the joint system ASE weakly couples with *universe* E'.
Interaction between SE is so dominant within the time scale T_{SE}.
However, for $t \gg T_{SE}$, a state on ASE is subjected to equibration.



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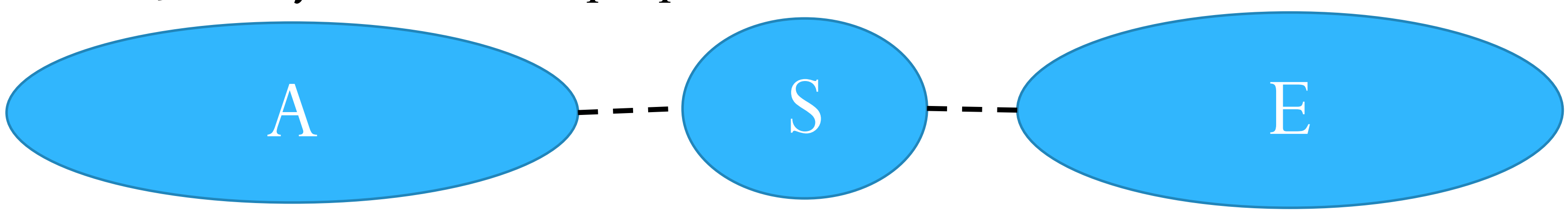


Our results:

We can "partially" identify completely unknown $|\Psi_{ASE}\rangle$ and H_{SE} so that E can be used for QC.

Actually, "partially" means "upto an equivalence class".

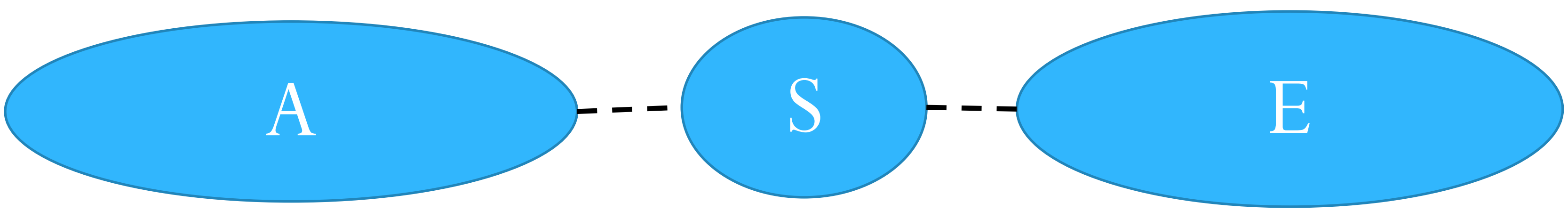
Equivalence of environments

Idea:

To use E as QC, we do not necessarily specify $|\Psi_{ASE}\rangle$ and H_{SE}.

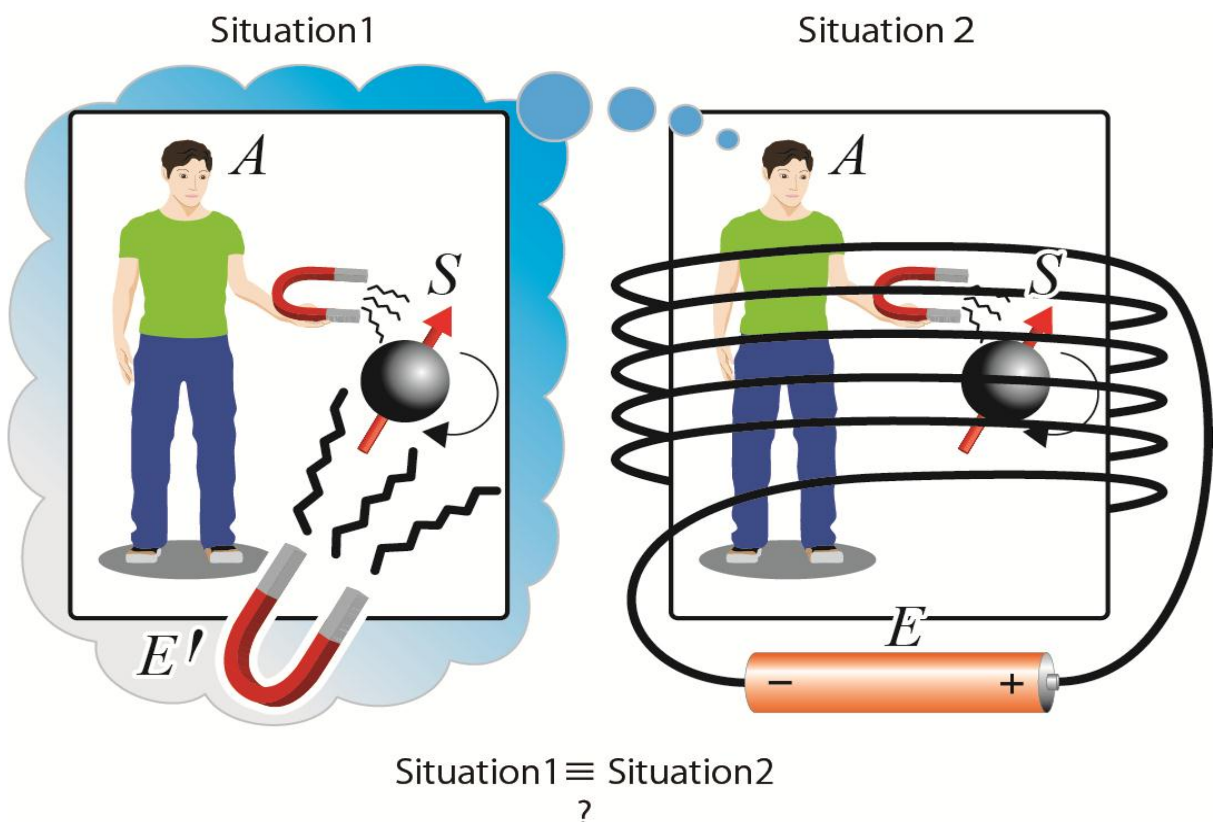
$(\mathrm{d}_{\mathrm{E}}, |\Psi_{ASE}\rangle, H_{SE})$ is enough to specify the environments.

However, even if $\mathrm{d}_{\mathrm{E}} \neq d'_E, |\Psi_{ASE}\rangle \neq |\Psi'_{ASE}\rangle$, and $H_{SE} \neq H'_{SE}$, $(\mathrm{d}_{\mathrm{E}}, |\Psi_{ASE}\rangle, H_{SE})$ and $(\mathrm{d}'_{\mathrm{E}}, |\Psi'_{ASE}\rangle, H'_{SE})$ may be equivalent as enviroments.



Equivalence of environments

We do not distinguish between E from E' as long as the experimenter cannot distinguish them.



Equivalence of environments

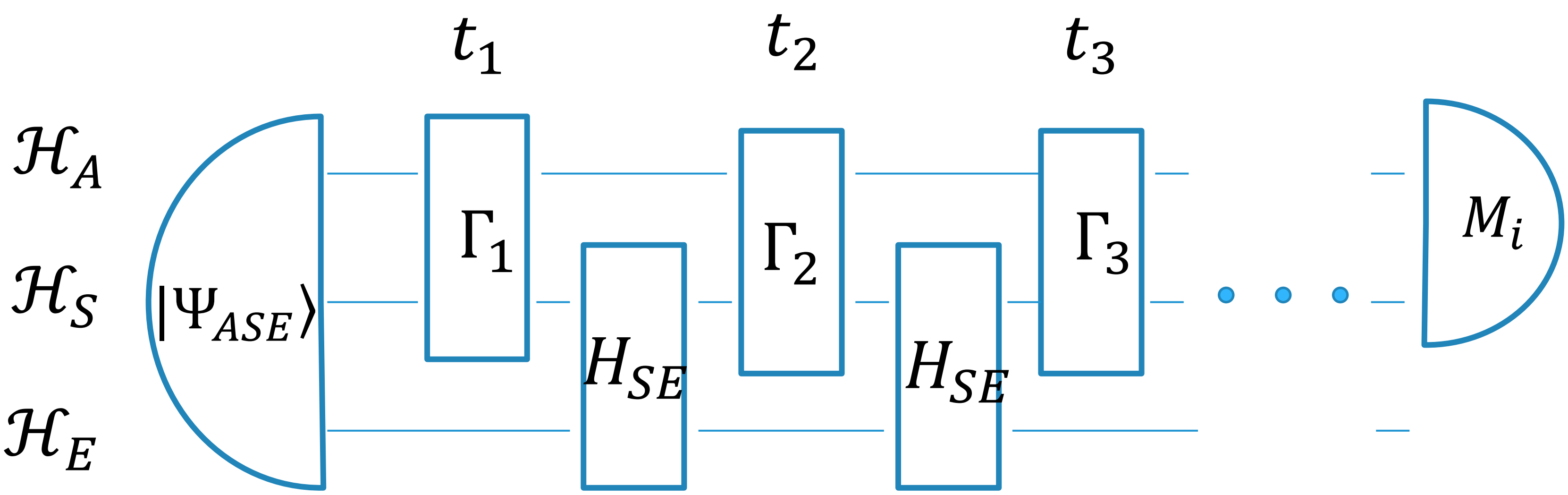
$(\mathrm{d_E}, |\Psi_{ASE}\rangle, H_{SE}) \equiv (\mathrm{d'_E}, |\Psi'_{ASE}\rangle, H'_{SE})$

Def ⟺ if there is no protocol on AS to distinguish them.

Protocol: Repeat the following 2 steps

1. Apply a non-deterministic quantum operation Γ_i on t_i
2. Freely evolute the system by H_{SE} from t_i to t_{i-1}.

Finally, measure A and S.

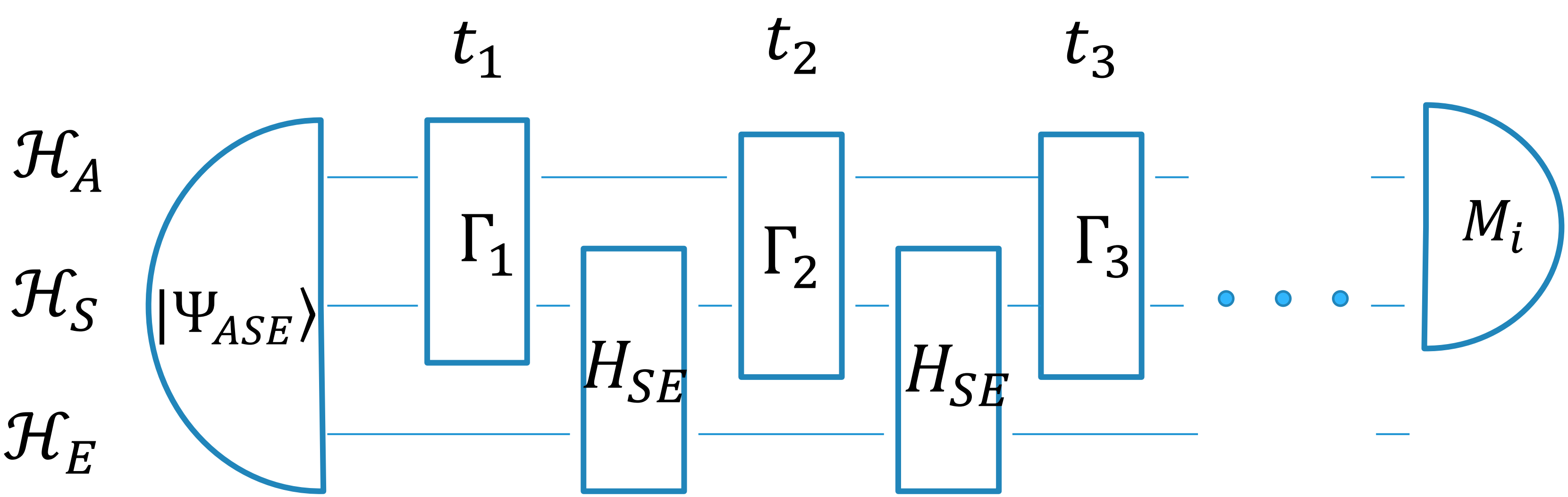


Equivalence of environments

$(\mathrm{d_E}, |\Psi_{ASE}\rangle, H_{SE}) \equiv (\mathrm{d'_E}, |\Psi'_{ASE}\rangle, H'_{SE})$

if there is no protocol on AS to distinguish them.

In this case, the results of any QC do not depend on whether the environment is $(\mathrm{d_E}, |\Psi_{ASE}\rangle, H_{SE})$ or $(\mathrm{d'_E}, |\Psi'_{ASE}\rangle, H'_{SE})$



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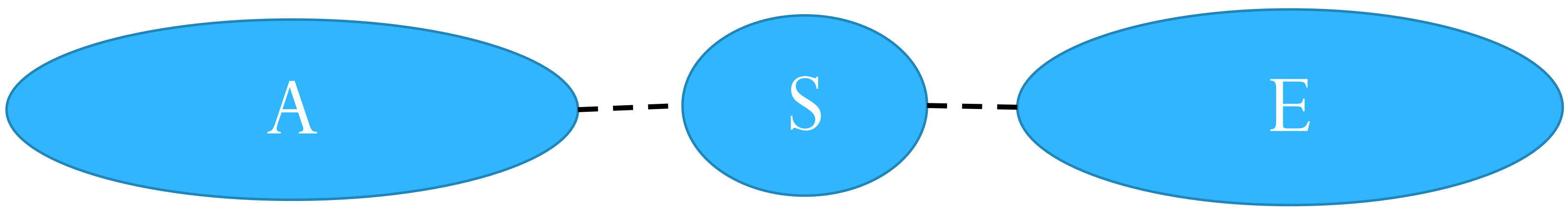


Our results:

We construct a protocol on AS to completely identify the equivalent class of $(\mathrm{d_E}, |\Psi_{ASE}\rangle, H_{SE})$, so that a part of E can be used for QC without A.

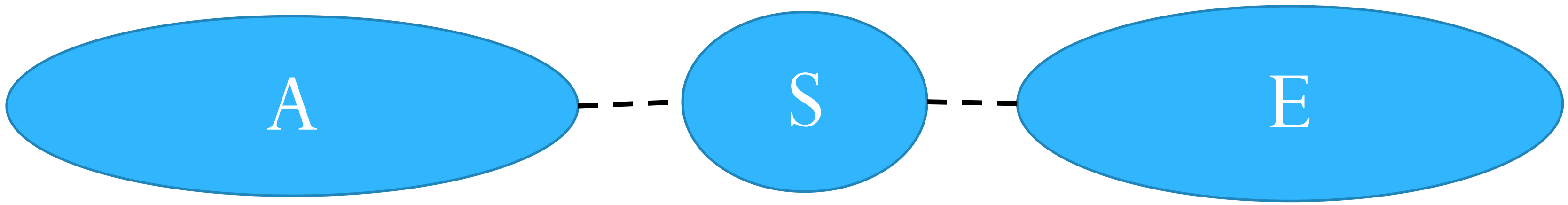
4. Protocol for tomography of env.

Protocol for Tomography of env.

The protocol consists of two stages:

1. Establishment of Entanglement between A and SE.

2. Tomography of the state on AS.



Protocol for Tomography of env.

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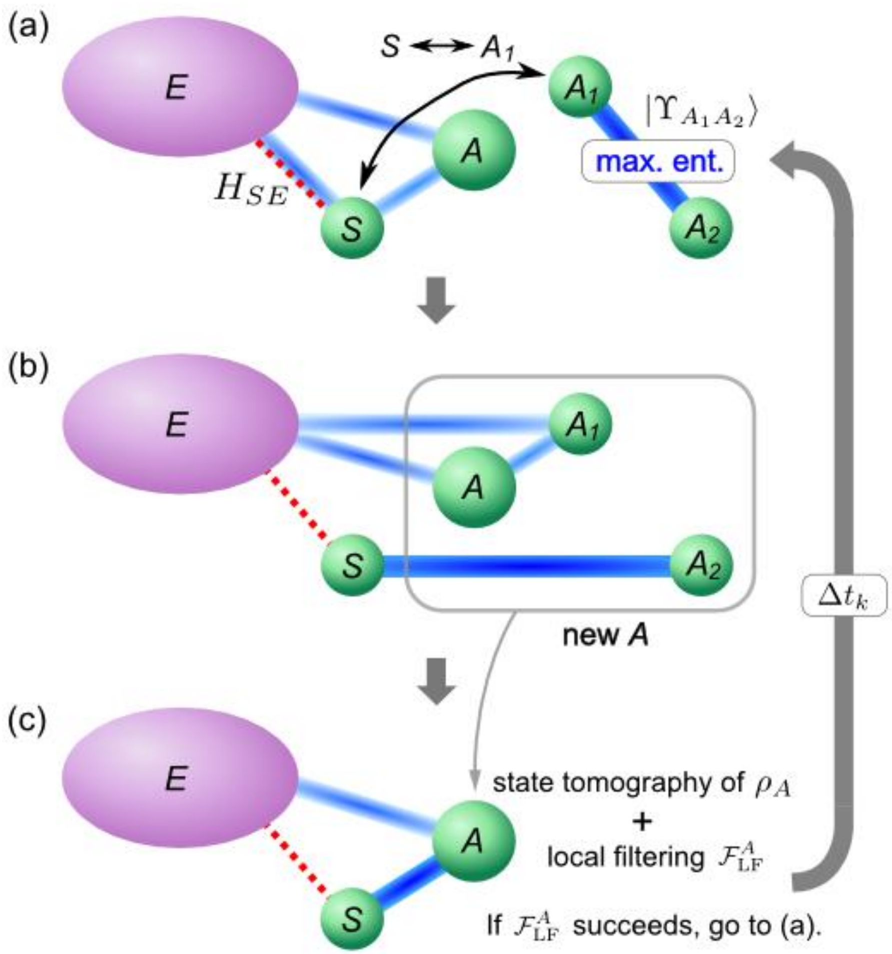


"State-steering protocol"

(a) Prepare a max. ent on ancilla, say $|\Upsilon_{A_1 A_2}\rangle$.

(b) Swap S and A_1

(c) State tomography of ρ_A,

and local filtering $\mathcal{F}^A_{LF}$

$$\mathcal{F}^A_{LF} := \sqrt{\lambda_{\min} \cdot \rho_A^{-1}}$$

We repeat (a), (b), (c) + time evolution for Δt_k.

Then, ent. between A and SE will saturate.

Protocol for Tomography of env.

Subroutine

(a) Prepare a max. ent on ancilla, say $|\Upsilon_{A_1A_2}\rangle$.

(b) Swap S and A_1

Entanglement gain $\Delta E(\rho_{AS})$:

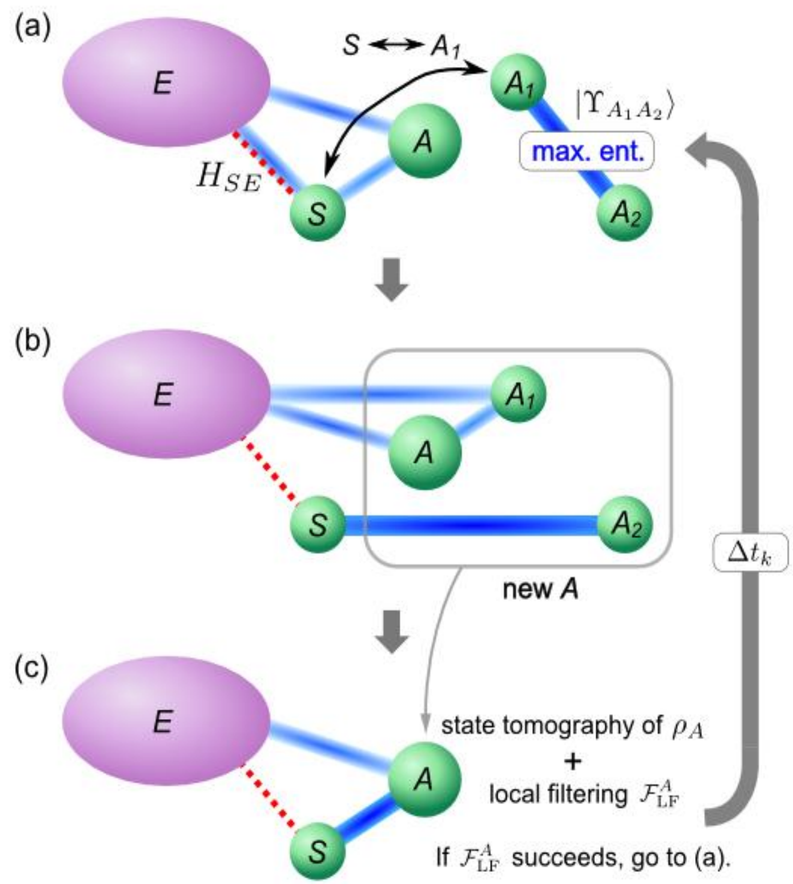
$$\Delta E(\rho_{AS}) \coloneqq S(\rho_{AS}) - S(\rho_A) + \log d_S$$

(c) State tomography of ρ_A,
filtering $\mathcal{F}_{LF}^A$;

$$\mathcal{F}_{LF}^A \coloneqq \sqrt{\lambda_{\min} \cdot \rho_A^{-1}}$$

After filtering ρ_A is a projector on its support.

(Schmidt coefficients of $|\Psi_{ASE}\rangle$ is "flat".)



Protocol for Tomography of env.

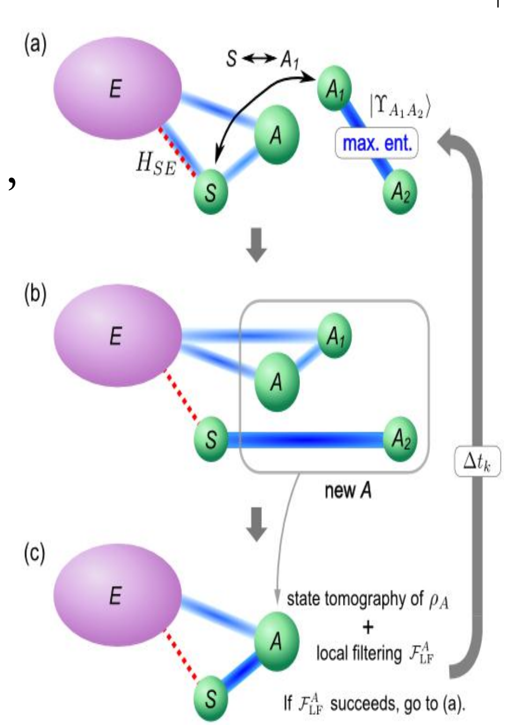
We stop the protocol <u>when ent. between A and SE saturates.</u>

$$\Delta E(\rho_{AS}) = 0$$

Since $\Delta E(\rho_{AS}) \coloneqq S(\rho_E) - S(\rho_{SE}) + \log d_S$
$= D(\rho_{SE} \,|\rho_S \otimes \rho_E) + D(\rho_S \,|\rho_{mix}),$

$\rho_{AS} = |\Upsilon_{SA_1}\rangle\langle\Upsilon_{SA_1}| \otimes \rho_{A_2}$ satisfying

- $\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$
- $|\Upsilon_{SA_1}\rangle$ is MES,
- ρ_{A_2} is a projector



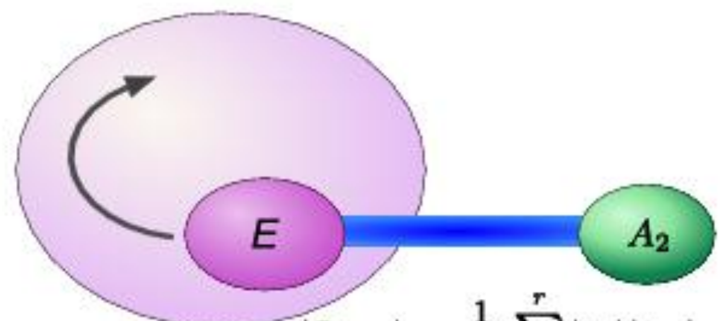
Protocol for Tomography of env.

$\rho_{AS} = |\Upsilon_{SA_1}\rangle\langle\Upsilon_{SA_1}| \otimes \rho_{A_2}$

with $\mathcal{H}_A = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$, $|\Upsilon_{SA_1}\rangle$ is MES, ρ_{A_2} is a projector.

That is, $|\Psi_{ASE}\rangle = |\Upsilon_{A_1S}\rangle \otimes |\Phi_{A_2E}\rangle$.

$|\Phi_{A_2E}\rangle$ may not be a MES.



However,

Possible to prove that $\exists(\mathrm{d}'_{\mathrm{E}}, |\Psi'_{ASE}\rangle, H'_{SE})$

s.t. $(\mathrm{d}_{\mathrm{E}}, |\Psi_{ASE}\rangle, H_{SE}) \equiv (\mathrm{d}'_{\mathrm{E}}, |\Psi'_{ASE}\rangle, H'_{SE})$

and $|\Psi_{ASE}\rangle = |\Upsilon_{A_1S}\rangle \otimes |\Upsilon_{A_2E}\rangle$ with $|\Upsilon_{A_2E}\rangle$ is MES.

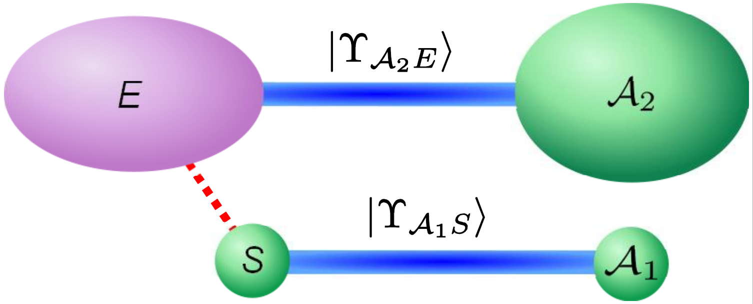
Protocol for Tomography of env.

2. Tomography of the state on AS

After the saturation,

$|\Psi_{ASE}\rangle = |\Upsilon_{A_1 S}\rangle \otimes |\Upsilon_{A_2 E}\rangle$, $|\Upsilon\rangle$ is MES.

upto equivalence of environments.

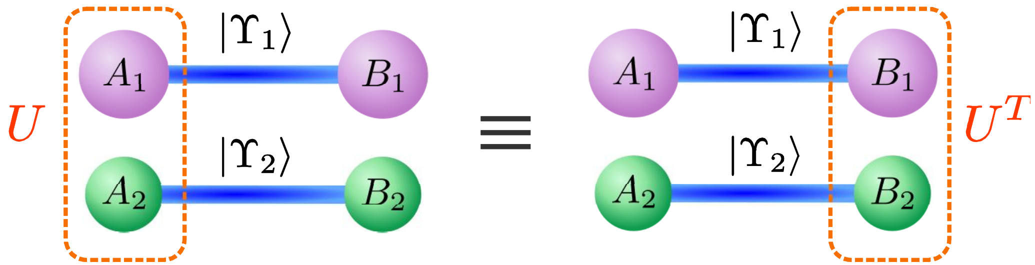


Since $I_A \otimes U|\Upsilon_{A_1 S}\rangle \otimes |\Upsilon_{A_2 E}\rangle = U^T \otimes I_{SE}|\Upsilon_{A_1 S}\rangle \otimes |\Upsilon_{A_2 E}\rangle$,

the evolution of AS:

$$i\frac{d}{dt}\rho_{AS} = [H_{SE}^T \otimes I_S, \rho_{AS}].$$

--(*)



Possible to prove all H_{SE} satisfies (*) is equivalent.

We only have to find H_{SE} satisfying (*) from tomography of $\rho_{AS}(t)$.

Protocol for Tomography of env.

Finding a Hermitian H_{SE} satisfying

$$i\frac{d}{dt}\rho_{AS}(t) = [H_{SE}^{T} \otimes I_S, \rho_{AS}(t)] \quad -(*)$$

$\exists\{\theta_\alpha\}_{\alpha=1}^{L}$ and $\{\rho_\alpha\}_{\alpha=0}^{L}$ s.t.

$$\rho_{AS}(t) = \rho_0 + \sum_{\alpha=1}^{L}\left(e^{i\theta_\alpha(t-t_0)}\rho_\alpha + e^{-i\theta_\alpha(t-t_0)}\rho_\alpha^{\dagger}\right).$$

We can determine $\{\theta_\alpha\}_{\alpha=1}^{L}$ and $\{\rho_\alpha\}_{\alpha=0}^{L}$ from an experimentally observed $\rho_{AS}(t)$, e.g. from at most the d_A^2-th derivative at t_0.

Protocol for Tomography of env.

Finding a Hermitian $\widetilde{H}_{SE}$ satisfying

$$i\frac{d}{dt}\rho_{AS}(t) = [\widetilde{H}_{SE}^{T} \otimes I_S, \rho_{AS}(t)] \quad -(*)$$

For given $\{\theta_\alpha\}_{\alpha=1}^{L}$ and $\{\rho_\alpha\}_{\alpha=0}^{L}$, Eq. (*) holds

if and only if H_{SE} satisfies

$$\begin{cases} [H_{SE}, \rho_0] = 0 \\ [H_{SE}, \rho_\alpha] = -\theta_\alpha \rho_\alpha, \qquad (1 \leq \alpha \leq L) \, . \end{cases} -(**)$$

Eq.(**) reduces a system of linear equations.

Solving the linear equations, we derive H_{SE}.

Tomography complete!

Protocol for Tomography of env.

The protocol consists of two stages:

1. State-steering protocol

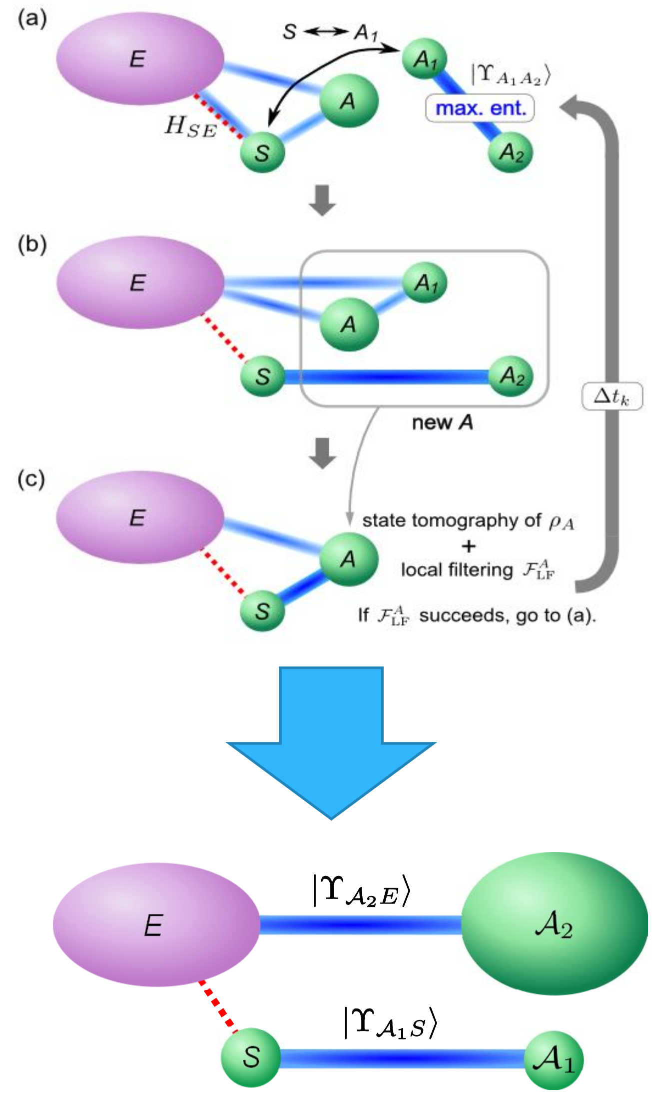
When the protocol halts,

A is maximally entangled with SE.

2. Tomography of the state on AS.

Tomography $\rho_{AS}(t)$

and solve linear equations.



Summary

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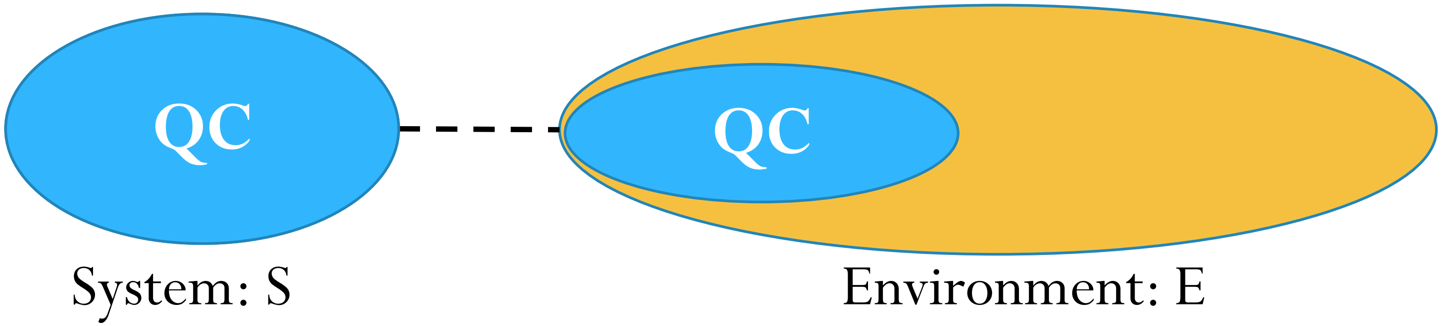


Part 2.
Mathematical Details

- Notations
- Equivalence of environments
- The maximally entanglement (ME) condition
- Tomography under the ME condition
- Protocol to achieve the ME condition

Notations

Projector onto $|\Psi\rangle$: $P(|\Psi\rangle) := |\Psi\rangle\langle\Psi|$

Hilbert space: $\mathcal{H}_{ASE} := \mathcal{H}_A \otimes \mathcal{H}_S \otimes \mathcal{H}_E$

$$\mathrm{d_A} := \dim \mathcal{H}_A \,, \mathrm{d_S} := \dim \mathcal{H}_S \,, \mathrm{d_E} := \dim \mathcal{H}_E$$

Interaction Hamiltonian: $H_{SE} \in \mathcal{B}(\mathcal{H}_{SE})$

Initial time: t_0 , Final time: t_∞ (we allow $t_\infty = +\infty$)

State on ASE at t_0 : $|\Psi_{ASE}\rangle$

after time evol. from t_0 to t : $|\Psi_{ASE}(t)\rangle := e^{-iH_{SE}(t-t_0)}|\Psi_{ASE}\rangle$

Environment during $[t_0, t_\infty)$ is characterized by a triple:

$(d_E, |\Psi_{ASE}\rangle, H_{SE})$

Equivalence of environment

Def: $(d_E, |\Psi_{ASE}\rangle, H_{SE}) \equiv \left(\tilde{d}_E, \left|\widetilde{\Psi}_{ASE}\right\rangle, \widetilde{H}_{SE}\right)$ in $[t_0, t_\infty)$,

if

$$\mathrm{Tr}_E\left(\Pi_i(\Gamma_i \otimes \mathcal{I}_E)\left(\mathcal{I}_A \otimes \mathcal{U}_{SE}^{(i)}\right)P(|\Psi_{\mathrm{ASE}}\rangle)\right)$$

$$=\mathrm{Tr}_E\left(\Pi_i(\Gamma_i \otimes \mathcal{I}_E)\left(\mathcal{I}_A \otimes \tilde{\mathcal{U}}_{SE}^{(i)}\right)P\left(|\widetilde{\Psi}_{\mathrm{ASE}}\rangle\right)\right)$$

$\forall n \in \mathbb{N}, \forall \{t_i\}_{i=1}^{n}$ with $t_0 < t_i < t_{i+1} < t_\infty$,

$\forall \left\{\mathcal{H}_{A_i}\right\}_{i=1}^{n}, \forall \{\Gamma_i\}_{i=1}^{n}$, where Γ_i is a CP and Trace non-increasing maps from $\mathcal{B}(\mathcal{H}_{A_i} \otimes \mathcal{H}_{A_{i+1}})$ to $\mathcal{B}(\mathcal{H}_{A_{i+1}} \otimes \mathcal{H}_S)$,

Where $\mathcal{U}_{SE}^{(i)}(\rho) := e^{-i(t_i - t_{i-1})H_{SE}}\rho\, e^{i(t_i - t_{i-1})H_{SE}}$

and $\tilde{\mathcal{U}}_{SE}^{(i)}(\rho) := e^{-i(t_i - t_{i-1})\widetilde{H}_{SE}}\rho\, e^{i(t_i - t_{i-1})H'_{SE}}$

Maximal Entanglement condition

Def: $(d_E, |\Psi_{ASE}\rangle, H_{SE})$ satisfies the maximal entanglement (ME) condition in $[t_0, t_\infty)$, if $\forall t \in [t_0, t_\infty)$, $\exists \mathcal{H}_{A_1}(t)$ and $\mathcal{H}_{A_2}(t)$, s.t. $\mathcal{H}_A = \mathcal{H}_{A_1}(t) \otimes \mathcal{H}_{A_2}(t)$, and

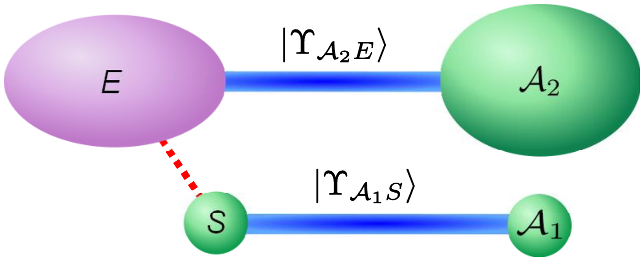
$$\mathrm{Tr}_E\big(P(|\Psi_{ASE}\rangle)\big) = P\big(|\Upsilon_{A_1S}(t)\rangle\big) \otimes \rho_A(t),$$

where $|\Upsilon_{A_1S}(t)\rangle$ is MES, and $\rho_A(t)$ is proportional to a projector.

The ME condition depends only on AS.

The ME condition is property of equiv. class of env.

$\rho_A(t)$ may no be a MES; rank $\rho_E < d_E$



The ME condition

Theorem 1: Suppose $(d_E, |\Psi_{ASE}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$.

Then, $(d_E, |\Psi_{ASE}\rangle, H_{SE}) \equiv \left(\tilde{d}_E, |\tilde{\Psi}_{ASE}\rangle, \tilde{H}_{SE}\right)$ in $[t_0, t_\infty)$,

if $\forall t \in [t_0, t_\infty)$,

$\mathrm{Tr}_E\left(P(|\Psi_{ASE}(t)\rangle)\right) = \mathrm{Tr}_E\left(P(|\tilde{\Psi}_{ASE}(t)\rangle)\right)$.

When the system satisfies the ME condition, an equivalence of natural time evolutions (without active operations) of AS is enough to show the equivalence of environments.

The ME condition

Corollary 1: Suppose $(d_E, |\Psi_{ASE}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$.

Then, $(d_E, |\Psi_{ASE}\rangle, H_{SE}) \equiv \left(\tilde{d}_E, \left|\widetilde{\Psi}_{ASE}\right\rangle, \widetilde{H}_{SE}\right)$ in $[t_0, t_\infty)$ implies that, for all $\tilde{t}_0$ and $\tilde{t}_\infty$ with $\tilde{t}_0 < \tilde{t}_\infty$,

$(d_E, |\Psi_{ASE}(t)\rangle, H_{SE}) \equiv \left(\tilde{d}_E, \left|\widetilde{\Psi}_{ASE}(t)\right\rangle, \widetilde{H}_{SE}\right)$ in $[\tilde{t}_0, \tilde{t}_\infty)$

If a system satisfies the ME condition, an equivalence of environments does not depend on an initial t_0 and a final time t_∞.

The ME condition

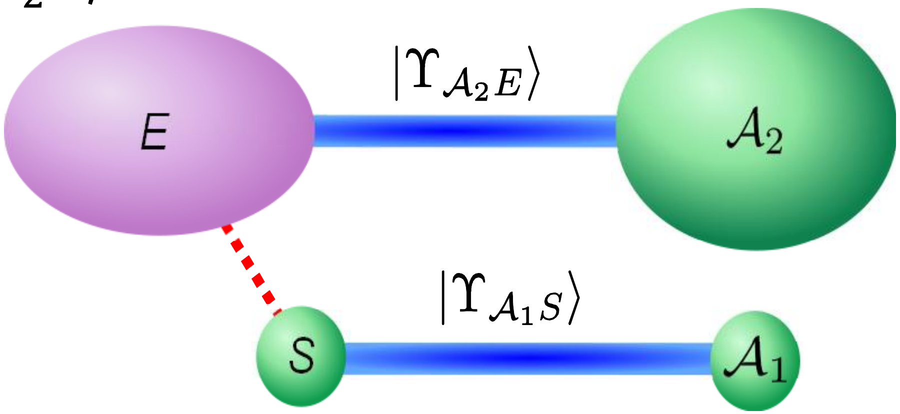
Theorem 2: Suppose $(d_E, |\Psi_{ASE}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$.

Then, $\exists\left(\tilde{d}_E, \left|\widetilde{\Psi}_{ASE}\right\rangle, \widetilde{H}_{SE}\right)$ s. t. $(d_E, |\Psi_{ASE}\rangle, H_{SE}) \equiv \left(\tilde{d}_E, \left|\widetilde{\Psi}_{ASE}\right\rangle, \widetilde{H}_{SE}\right)$ in $[t_0, t_\infty)$,

and $\left|\widetilde{\Psi}_{ASE}\right\rangle$ is MES.

Thus, $\left|\widetilde{\Psi}_{ASE}\right\rangle = \left|\Upsilon_{A_1S}\right\rangle \otimes \left|\Upsilon_{A_2E}\right\rangle$;

$\left|\Upsilon_{A_2E}\right\rangle = \frac{1}{r}\sum_{i=1}^{r} |ii\rangle$.



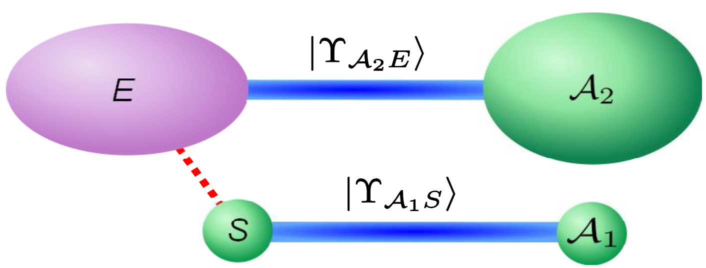
The ME condition

Corollary 2: Suppose $(d_E, |\Psi_{ASE}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$.

Then, for all $\tilde{t}_0$ and $\tilde{t}_\infty$ with $\tilde{t}_0 < \tilde{t}_\infty$, $(d_E, |\Psi_{ASE}(t)\rangle, H_{SE})$ satisfies the ME condition in $[\tilde{t}_0, \tilde{t}_\infty)$

The ME condition does not depend on an initial time t_0 and a final time t_∞.

If the system satisfies the ME condition for a short time, the system will always satisfy the ME condition after that.



Tomography under the ME condition

Corollary 3: Suppose $(d_E, |\Psi_{ASE}\rangle, H_{SE})$ satisfies the ME condition in $[t_0, t_\infty)$, and $|\Psi_{ASE}\rangle$ can be written down as

$$\left|\widetilde{\Psi}_{ASE}\right\rangle = \left|\Upsilon_{A_1 S}\right\rangle \otimes \left|\Upsilon_{A_2 E}\right\rangle .$$

Then, if a Hermitian matrix $\widetilde{H}_{SE}$ on $\mathcal{H}_S \otimes \mathcal{H}_E$ satisfies

$$i\frac{d}{dt}\rho_{AS}(t) = \left[\widetilde{H}_{SE}^T \otimes I_S, \rho_{AS}(t)\right] \quad -(*)$$

For all t in an neighbourhood of t_0,

then, $(d_E, |\Psi_{ASE}\rangle, H_{SE}) \equiv \left(d_E, |\Psi_{ASE}\rangle, \widetilde{H}_{SE}\right)$ in $[t_0, t_\infty)$

For an experimentally observed $\rho_{AS}(t)$,

All $\widetilde{H}_{SE}$ satisfying Eq.(*) can be used as a interaction Hamiltonian.

Tomography under the ME condition

Finding a Hermitian $\widetilde{H}_{SE}$ satisfying

$$i\frac{d}{dt}\rho_{AS}(t) = \left[\widetilde{H}_{SE}^{T} \otimes I_S, \rho_{AS}(t)\right] \quad -(*)$$

$\exists\{\theta_\alpha\}_{\alpha=1}^{L}$ and $\{\rho_\alpha\}_{\alpha=0}^{L}$ s.t.

$$\rho_{AS}(t) = \rho_0 + \sum_{\alpha=1}^{L}\left(e^{i\theta_\alpha(t-t_0)}\rho_\alpha + e^{-i\theta_\alpha(t-t_0)}\rho_\alpha^{\dagger}\right).$$

We can determine $\{\theta_\alpha\}_{\alpha=1}^{L}$ and $\{\rho_\alpha\}_{\alpha=0}^{L}$ from an experimentally observed $\rho_{AS}(t)$, e.g. from at most the d_A^2-th derivative at t_0.

Tomography under the ME condition

Finding a Hermitian $\widetilde{H}_{SE}$ satisfying

$$i\frac{d}{dt}\rho_{AS}(t) = [\widetilde{H}_{SE}^{T} \otimes I_S, \rho_{AS}(t)] \quad -(*)$$

We can prove: For given $\{\theta_\alpha\}_{\alpha=1}^{L}$ and $\{\rho_\alpha\}_{\alpha=0}^{L}$, if H_{SE} satisfy

$$\begin{cases} [H_{SE}, \rho_0] = 0 \\ [H_{SE}, \rho_\alpha] = -\theta_\alpha \rho_\alpha, \quad (1 \leq \alpha \leq L) \end{cases} , -(**)$$

then, Eq. (*) holds.

Eq.(**) reduces a system of linear equations.

Solving the linear equations, we derive H_{SE}.

Protocol to achieve the ME condition

Remarks

- $d_A = 1$ at the beginning of the protocol.
- At the beginning ($t = 0$), the joint system SE is in a pure state.
- We always need to derive the present description of the state on AS. So, we need to apply a state-tomography in every small time period on AS. Thus, we need to repeat each step of our protocol infinitely many times.
- The time t is the elapsed-time. Thus, we restart the clock from $t = 0$ when a non-deterministic operation is failed, or we apply the state-tomography.

Protocol to achieve the ME condition

Subroutine

(a) Prepare a max. ent on ancilla, say $|\Upsilon_{A_1A_2}\rangle$.

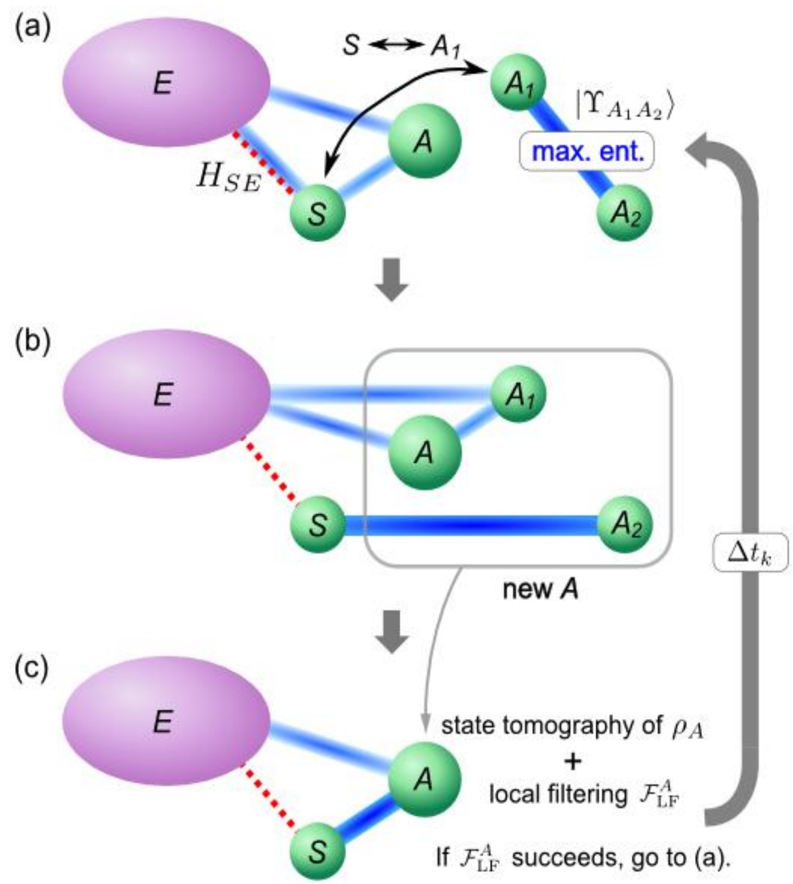
(b) Swap S and A_1

Entanglement gain $\Delta E(\rho_{AS})$:

$$\Delta E(\rho_{AS}) := S(\rho_{AS}) - S(\rho_A) + \log d_S$$

(c) State tomography of ρ_A, filtering $\mathcal{F}_{LF}^A$;

$$\mathcal{F}_{LF}^A := \sqrt{\lambda_{\min} \cdot \rho_A^{-1}}$$



Protocol to achieve the ME condition

Protocol with parameter Δt:

Step 0: At the beginning, say $t = 0$, there is no ancillary system A, that is $\dim\mathcal{H}_A = 1$, and we set counter $C = 0$.

Step 1: Increment the counter C by one. Implement the subroutine. Define this time as t_C.

Step 2: Define ϵ_C as

$$\epsilon_C := \frac{1}{2}\sup\{\Delta E(t)|t \in [t_C, t_C + \Delta t]\}$$

by means of the state tomography on AS during $[t_C, t_C + \Delta t]$.

Stop the protocol if $\epsilon_C = 0$, otherwise let the system SE evolves for a time while $\Delta E(t) < \epsilon_C$, and go back to the step 1, when $\Delta E(t) = \epsilon_C$.

Protocol to achieve the ME condition

Theorem 3. The protocol halts at latest when $C = d_S d_E$, or equivalently $t = d_S d_E \Delta t$.

In other words, $\exists \mathrm{K} \leq d_E$ s.t. $\Delta E(t) = 0$ for all $\mathrm{t} \in [t_C, t_C + \Delta t]$.

∵) After the application of the last subroutine, ρ_A is proportional to a projector.

If $\exists \mathrm{t} \in [t_C, t_C + \Delta t]$ s.t. $\Delta E(t) > 0$, then $\mathrm{rank}\, \rho_A$ increments by one by a swap operation.

Thus, $\mathrm{C} \leq \mathrm{rank}\, \rho_A = \mathrm{rank}\, \rho_{SE} \leq d_S d_E$.

Protocol to achieve the ME condition

Theorem 4. Suppose the counter $C = K$ when the protocol halts. Then, for all t_∞ satisfying $t_\infty > t_K$, $(d_E, |\Psi_{ASE}(t_K)\rangle, H_{SE})$ satisfies the ME condition.

$|\Psi_{ASE}(t_K)\rangle$: a state just after K-th implementation of the subroutine

$$\because)\ \Delta E(\rho_{AS}) := S(\rho_E) - S(\rho_{SE}) + \log d_S$$
$$= D(\rho_{SE} | \rho_S \otimes \rho_E) + D(\rho_S | \rho_{mix}).$$

Thus, $\Delta E(\rho_{AS})$=0, iff $\rho_{SE} = \rho_S \otimes \rho_E$ and $\rho_S = \rho_{mix}$.

This implies the ME condition.

Therefore, we succeeded to achieve the ME condition.

So, tomography of environments is possible.

Summary

Our problem settings

1. $\dim \mathcal{H}_S < +\infty$ and $\dim \mathcal{H}_E < +\infty$
2. An arbitrarily large ancilla (the system A) is available.
3. The joint system can be prepared in $|\Psi_{ASE}\rangle$ at an initial time t_0.

Our results:

We construct a protocol on AS to completely identify the equivalent class of $(\mathrm{d_E}, |\Psi_{ASE}\rangle, H_{SE})$, so that a part of E can be used for QC without A.