Extension of the Watanabe-Sagawa-Ueda uncertainty relations to infinite-dimensional systems

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Introduction: The Watanabe-Sagawa-Ueda Uncertainty Relations

Watanabe, Sagawa, and Ueda[1,2] proposed definitions for the measurement errors and disturbance to observables in quantum measurements, based on the quantum estimation theory.

- The parametrization:

$$\rho(\boldsymbol{\theta}) = \frac{1}{d}I + \sum_{i=1}^{d^2-1}\theta_i\lambda_i =: \frac{1}{d}I + \boldsymbol{\theta}\cdot\boldsymbol{\lambda}$$

- The measurement error:

$$\varepsilon(A;\rho,M) := \begin{cases} \boldsymbol{a}\cdot(J_{\boldsymbol{\theta}}^M)^+\boldsymbol{a} - \sigma_{\boldsymbol{\theta}}(A)^2 & (\boldsymbol{a} := \nabla_{\boldsymbol{\theta}}\langle A\rangle_{\boldsymbol{\theta}} \in (\ker J_{\boldsymbol{\theta}}^M)^\top) \\ +\infty & (\text{otherwise}) \end{cases}$$

- The disturbance:

$$\eta(A;\rho,\mathcal{E}) := \begin{cases} \boldsymbol{a}\cdot(J_{\mathcal{E}\circ\rho(\boldsymbol{\theta})}^{\mathrm{S}})^+\boldsymbol{a} - \sigma_{\boldsymbol{\theta}}(A)^2 & \left(\boldsymbol{a} \in (\ker J_{\mathcal{E}\circ\rho(\boldsymbol{\theta})}^{\mathrm{S}})^\top\right) \\ +\infty & (\text{otherwise}) \end{cases}$$

- The error-error uncertainty relation:

$$\varepsilon(A;\rho,M)\varepsilon(B;\rho,M) \geq \frac{1}{4}\left|\langle[A,B]\rangle_\rho\right|^2$$

- The error-disturbance uncertainty relation:

$$\varepsilon(A;\rho,\mathcal{I})\eta(B;\rho,\mathcal{I}) \geq \frac{1}{4}\left|\langle[A,B]\rangle_\rho\right|^2$$

Classical Estimation For Infinite-Dimensional Parameters

We consider the case where the probability measure P_θ on the measurable space $(\Omega,\mathfrak{B})$ is characterized by the parameter $\theta\in\Theta\subset\Xi$, where Ξ is a real Hilbert space.

- The logarithmic dereivative of P_θ in the direction $\phi\in\Xi$ [3]:

$$l_\theta(x;\phi) := \frac{dD_\theta P_\theta(\cdot)[\phi]}{dP_\theta}(x)$$

- The Fisher information operator $J_\theta\in\mathcal{L}(\Xi)$:

$$(\phi, J_\theta\chi)_\Xi = F(\phi,\chi) := \mathbb{E}_\theta\left[l_\theta(X;\phi)l_\theta(X;\chi)\right]$$

- The Cramér-Rao inequality:

$$\mathbb{V}\mathrm{ar}_\theta[\hat{\boldsymbol{f}}] \geq \left(\nabla_\theta\bar{\boldsymbol{f}}, J_\theta^+(\nabla_\theta\bar{\boldsymbol{f}})^\top\right)_\Xi,$$

  where $\bar{\boldsymbol{f}} := \mathbb{E}_\theta[\hat{\boldsymbol{f}}]$.

- Monotonicity under Markov maps:

$$J_{P_\theta} \geq J_{TP_\theta}$$

Quantum Estimation For Infinite-Dimensional Parameters

Suppose that the density operator ρ_θ is characterized by the parameter $\theta\in\Theta\subset\Xi$, where Ξ is a real Hilbert space. Let J_θ^M denote the Fisher information operator associated with the probability measure $P_\theta^M(\cdot) := \mathrm{Tr}[\rho(\theta)M(\cdot)]$, where M is a positive operator-valued measure (POVM).

- $f:\mathbb{R}_+\to\mathbb{R}$: an operator monotone function:

$$A\leq B\in\mathcal{L}(\mathcal{H}) \quad\Rightarrow\quad f(A)\leq f(B)\in\mathcal{L}(\mathcal{H})$$

- A map between the space of operators on the Hilbert space $\mathcal{H}$:

$$\mathbf{K}_\rho^f := \mathbf{R}_\rho^{1/2}f(\mathbf{L}_\rho\mathbf{R}_\rho^{-1})\mathbf{R}_\rho^{1/2},$$

  where

$$\mathbf{L}_\rho X := \rho X, \quad \mathbf{R}_\rho X := X\rho.$$

- The f-logarithmic derivative in the direction $\phi\in\Xi$:

$$L_\theta^f(\phi) := (\mathbf{K}_\rho^f)^{-1}D_\theta\rho[\phi]$$

- The f-quantum Fisher information operator $J_\theta^f\in\mathcal{L}(\Xi)$:

$$\left(\phi, J_\theta^f\chi\right)_\Xi = H_\theta^f(\phi,\chi) := \left(\mathbf{K}_{\rho(\theta)}^f L_\theta^f(\phi), L_\theta^f(\chi)\right)_{\mathrm{HS}}$$

  (When H_θ^f is $\mathbb{C}$-valued, we extend H_θ^f to operate on $\tilde{\Xi} := \Xi + \mathrm{i}\Xi$.)

- The quantum Cramér-Rao inequality:

$$J_\theta^f \geq J_\theta^M$$

- Monotonicity under quantum channels:

$$J_{\rho(\theta)}^f \geq J_{\mathcal{E}(\rho(\theta))}^f$$

Main Results

Let us consider the estimation of the expectation value $\langle A\rangle_\theta$ of an observable A in an unknown state $\rho(\theta)$ of a quantum system represented by an infinite-dimensional Hilbert space $\mathcal{H}$.

- The parametrization:

$$\rho(\theta) = \rho_0 + \theta, \quad \theta\in\Theta\subset\Xi,$$

  where ρ_0 is a fixed density operator and

$$\Xi := \{A\in\mathcal{T}(\mathcal{H}) \mid \mathrm{Tr}\,A = 0, A = A^*\},$$

  which is a real Hilbert space with the inner product

$$(A,B)_\Xi := \mathrm{Tr}[A^*B], \quad A,B\in\Xi.$$

- The f-correlation function:

$$\mathcal{C}_\rho^f(A,B) := \mathrm{Tr}[A^*\mathbf{K}_\rho^f B] - \mathrm{Tr}[\rho A]^*\,\mathrm{Tr}[\rho B]$$

- The inverse of the f-quantum Fisher information operator:

$$(J_{\rho(\theta)}^f)^{-1} = C_{\rho(\theta)}^f,$$

  where $C_\rho^f\in\mathcal{L}(\Xi)$ is the operator defined by

$$\left(\phi, C_\rho^f\psi\right)_\Xi = \mathcal{C}_\rho^f(\phi,\psi), \quad \phi,\psi\in\Xi.$$

- The measurement error:

$$\varepsilon(A;\rho(\theta),M) := \begin{cases} \left(a, (J_{\rho(\theta)}^M)^+a\right)_\Xi - \sigma_\theta(A)^2 & \left(a := \nabla_\theta\langle A\rangle_\theta \in (\ker J_{\rho(\theta)}^M)^\perp\right) \\ +\infty & (\text{otherwise}) \end{cases}$$

- The disturbance:

$$\eta(A;\rho(\theta),\mathcal{E}) := \begin{cases} \left(a, (J_{\mathcal{E}\circ\rho(\theta)}^{\mathrm{S}})^+a\right)_\Xi - \sigma_\theta(A)^2 & \left(a \in (\ker J_{\mathcal{E}\circ\rho(\theta)}^{\mathrm{S}})^\perp\right) \\ +\infty & (\text{otherwise}) \end{cases}$$

- A tighter error-error uncertainty relation:

$$\varepsilon(A;\rho(\theta),M)\varepsilon(B;\rho(\theta),M) \geq \mathcal{R}_M(A,B)^2 + \left|\frac{1}{2}\langle[A,B]\rangle_\theta\right|^2,$$

  where

$$\mathcal{R}_M(A,B) := \left(\nabla_\theta\langle B\rangle_\theta, (J_{\rho(\theta)}^M)^+\nabla_\theta\langle A\rangle_\theta\right)_\Xi - \mathcal{C}_\theta^{\mathrm{S}}(A,B).$$

  The proof follows from the Cauchy-Schwarz inequality with respect to the semi-inner product defined by

$$\langle\!\langle A,B\rangle\!\rangle := \left(\nabla_\theta\langle B\rangle_\theta, (J_{\rho(\theta)}^M)^+\nabla_\theta\langle A\rangle_\theta\right)_\Xi - \mathcal{C}_\theta^{\mathrm{R}}(A,B).$$

- A tighter error-disturbance uncertainty relation: There exists a POVM M such that

$$\varepsilon(A;\rho(\theta),\mathcal{I})\eta(B;\rho(\theta),\mathcal{I}) \geq \mathcal{R}_M(A,B)^2 + \left|\frac{1}{2}\langle[A,B]\rangle_\theta\right|^2$$

Conclusion

- The definitions of measurement error and disturbance by Watanabe, Sagawa, and Ueda can be extended to infinite-dimensional quantum systems.
- In the process, we constructed classical and quantum estimation theories for infinite-dimensional parameters in a way that naturally extends the finite-dimensional case.
- We found the inverse of the f-quantum Fisher information operator J_θ^f when the parametrization is $\rho(\theta) = \rho_0 + \theta$.
- We obtained stricter inequalities than the original ones.

References

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